

## Intersecting D7-branes, I5-branes and conifolds

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**ABSTRACT:** A candidate supergravity solution of intersecting D7-branes with a five-dimensional intersecting domain (an I5-brane) is presented. This displays an enhanced Poincare symmetry and supersymmetry away from the brane cores. We also explore the possibility of a relation between the intersection region of D7-branes and conifolds through F-theory.

**KEYWORDS:** F-Theory, D-branes, Intersecting branes models.

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## 1. Introduction

The study of different configurations of branes provides us with information about their dynamics and properties of the gauge theories that reside in them. For example, by considering space filling intersecting branes as in [1], the anomaly inflow mechanism determines the coupling of D branes to Ramond-Ramond forms. As is noted in [1] the two distinct

supersymmetric configurations of space filling intersecting branes up to T-duality are intersecting D5-branes over  $1 + 1$  dimensions ( $I1$  brane) and intersecting D7-branes over  $5 + 1$  dimensions ( $I5$  brane). The characteristic feature of such configuration is that they describe chiral theories in the intersection domain. There is a curious symmetry enhancement in the  $I1$  observed in [2], where it was found that Poincare symmetry is enhanced from  $SO(1, 1)$  to  $SO(1, 2)$ , and the number of supersymmetries was also doubled. This phenomenon was confirmed both from the weakly coupled gauge theory picture and the supergravity limit. It is thus interesting to see if this also occurs in the  $I5$ -brane. However in this case the microscopic description in terms of six-dimensional gauge theory is problematic. Furthermore the validity of the use of DBI action in analysing the D7-branes dynamics is not well understood, since an angle deficit is created by every 7-brane independent of the value of the string coupling. On the other hand it should be possible to study the supergravity limit of this configuration. This is one of the main purposes of this paper.

Our solutions generally describe a stack of  $N_1$  coincident D7-branes intersecting another stack of  $N_2$  coincident D7-branes in an  $I5$  configuration, where  $N_1, N_2 \leq 24$ . Most of our discussion will be restricted to the case with equal numbers of branes,  $N_1 = N_2$ . The solution is based on a generalisation of the stringy cosmic string [3]. Interestingly, the solutions also have enhanced Poincare symmetry and supersymmetry, as in the case of  $I1$  branes. However, in this case the enhancement is a property of the full solution rather than the near-horizon limit. The solutions also generalise to situations in which branes are separated in one of the stacks and also to branes intersecting at angles.

The second part of this paper explores possible connections between intersecting D7-branes and the conifold singularity of elliptically fibred Calabi-Yau manifolds. The fact that a particular system of branes should be dual to special spacetime geometry has been discussed at length in the literature (e.g. [4]). The explicit example of intersecting NS5-branes which T-dualise to the conifold provides an explanation for gauge symmetry enhancement as the complex moduli of the geometry is tuned. The similarities between these NS5 configurations and intersecting D7-branes suggests an extension of these ideas might be possible. In particular, F-theory describes elliptically fibred Calabi-Yau manifolds in terms of two intersecting stack of D7-branes with 24 branes in each stack. We will show explicitly that the conifold arises in the limit in which two D7-branes, one from each stack, intersect each other but are far from the others. However, we have not been able to obtain the conifold metric from our solution, mainly because of the fact that F-theory distinguishes the extra fibred torus, which is unphysical, from the rest of the spacetime, thus breaking the  $SO(4)$  symmetry required of a conifold.

The organisation of the paper is as follows: The solution for parallel D7-branes and its origin from the stringy cosmic string solution will provide the framework for the intersecting brane solution. The stringy cosmic string [3] and its connection to D7-branes and F-theory, which foreshadows the analysis of the conifold, are reviewed in the appendix. In section 2 and 3 we shall present the candidate solutions of the intersecting D7-branes and some of their properties. In section 4 we will discuss the possibility of obtaining a conifold at the intersection. Further comments will be made in section 5.

## 2. Intersecting D7-brane solutions of the Einstein equations

The bosonic action and the equations of motion are as discussed in appendix (A.1) To obtain a solution that represents a system of two sets of D7-branes intersecting in six spacetime dimensions with four relatively transverse directions, we assume that the metric takes the following form:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + K_{a\bar{b}} dz^a d\bar{z}^b \quad (2.1)$$

where  $\alpha, \beta \in \{0, 1, 2, 3, 4, 5\}$  and  $a, b \in \{1, 2\}$  such that  $z^a$ 's are two complex dimensions. In other words, the four relatively transverse directions form a complex two dimensional Kahler manifold whose Kahler metric is given by  $K_{a\bar{b}} = \partial_a \bar{\partial}_b K$ , where  $K$  is the Kahler Potential.

The  $\tau$  equation (A.3) can be solved if  $\tau$  is a holomorphic function of  $z^a$ . Then the Einstein equation (A.4) is simplified to

$$R_{a\bar{b}} = -\partial_a \bar{\partial}_b \ln \det K_{a\bar{b}} = \frac{1}{4\tau_2^2} (\partial_a \tau \bar{\partial}_b \bar{\tau}) = -\partial_a \bar{\partial}_b \ln \tau_2 \quad (2.2)$$

where we have made use of the property of a Kahler manifold in the first equality.

The equation is solved by

$$\tau = \tau(z^a), \quad (2.3)$$

$$\det K_{a\bar{b}} = |F(z^a)|^2 \tau_2 \quad (2.4)$$

where  $F(z^a)$  is any arbitrary holomorphic function.

SUSY transformations of the dilatino  $\lambda$  and gravitinos  $\psi_\mu$  are given by:

$$\delta\lambda = -\frac{1}{2\tau_2} (\partial_a \tau e_A^a \Gamma^A + \partial_{\bar{a}} \tau e_A^{\bar{a}} \Gamma^{\bar{A}}) \epsilon^* \quad (2.5)$$

$$\delta\psi_a = D_a \epsilon = \left( \partial_a + \frac{1}{4} w_a^{AB} \Gamma_{[AB]} + \frac{i}{4} \frac{\partial_a \tau_1}{\tau_2} \right) \epsilon \quad (2.6)$$

where  $w_a^{AB}$  are the spin connections (whose non-trivial components are along the relatively transverse directions only) and  $A, B \in \{1, 2\}$  are tangent space indexes. Both variations vanish when we substitute in the solution (2.3), (2.4) and only one quarter of maximal supersymmetries are preserved.

### 2.1 An ansatz for coincident D7-branes intersecting orthogonally

To obtain an exact form of the metric, we make further assumptions of the form of the holomorphic function  $F$  and  $\tau_2$ : suppose we are having  $N_1$  coincident D7-branes at the origin transverse to directions  $z$  intersecting another set of  $N_2$  D7-branes also at the origin transverse to directions  $w$ , then we propose that

$$j(\tau) = \frac{1}{z^{N_1} w^{N_2}}, \quad (2.7)$$

$$F(z, w) = \eta^2(\tau) \left( z^{-\frac{N_1}{12}} w^{-\frac{N_2}{12}} \right). \quad (2.8)$$

This proposal is a direct generalisation of the solution for parallel D7-branes. The solution restricts  $\tau$  in the fundamental region and ensures modular invariance of the metric. The factor  $z^{-\frac{N_1}{12}} w^{-\frac{N_2}{12}}$  cancels the divergence in  $\eta^2(\tau)$  as  $z$  and  $w$  approach 0.

In this proposal, both  $F$  and  $\tau$  depend only on  $z^{N_1} w^{N_2}$  and this suggests that we should define new variables:

$$y = -(N_1 \ln z + N_2 \ln w), \quad x = (N_2 \ln z - N_1 \ln w). \quad (2.9)$$

In these new variables, equation (2.4) becomes

$$\det K_{z\bar{w}} = \left| \begin{array}{cc} -\frac{N_1}{z} & -\frac{N_2}{w} \\ \frac{N_2}{z} & -\frac{N_1}{w} \end{array} \right|^2 (K_{x\bar{x}} K_{y\bar{y}} - K_{y\bar{x}} K_{x\bar{y}}) = |F|^2 \tau_2. \quad (2.10)$$

Rearranging gives

$$(K_{x\bar{x}} K_{y\bar{y}} - K_{y\bar{x}} K_{x\bar{y}}) = \frac{\exp\left(\frac{N_2 - N_1}{N_1^2 + N_2^2}(x + \bar{x})\right) \exp\left(-\frac{N_2 + N_1}{N_1^2 + N_2^2}(y + \bar{y})\right)}{(N_1^2 + N_2^2)^2} |F|^2 \tau_2, \quad (2.11)$$

using the fact that

$$|zw|^2 = \exp\left(\frac{N_2 - N_1}{N_1^2 + N_2^2}(x + \bar{x})\right) \exp\left(-\frac{N_2 + N_1}{N_1^2 + N_2^2}(y + \bar{y})\right). \quad (2.12)$$

The simplest solution would be to require that the metric be block diagonal in  $x$  and  $y$  i.e.  $K_{x\bar{y}} = K_{y\bar{x}} = 0$  such that the Kahler Potential takes the form

$$K = f(x, \bar{x}) + g(y, \bar{y}). \quad (2.13)$$

The resultant metric is

$$K_{x\bar{x}} = \alpha \exp\left(\frac{N_2 - N_1}{N_1^2 + N_2^2}(x + \bar{x})\right) \quad (2.14)$$

$$K_{y\bar{y}} = \beta \exp\left(-\frac{N_2 + N_1}{N_1^2 + N_2^2}(y + \bar{y})\right) |F|^2 \tau_2, \quad (2.15)$$

where  $\alpha, \beta$  are constants satisfying  $\alpha\beta = 1/(N_1^2 + N_2^2)^2$ . The choice of  $\alpha$  and  $\beta$  has the effect of altering the way the arguments of  $z, w$  are mixed with each other. Consider for example the case where  $N_1 = N_2 = 12$  where both  $K_{x\bar{x}}$  and  $K_{y\bar{y}}$  become flat. The compact part of the line element in terms of the original coordinate is (i.e. for  $z = |z| \exp(i\theta)$  and  $w = |w| \exp(i\phi)$ )

$$ds_{\theta, \phi}^2 = \alpha d(\theta - \phi)^2 + \beta d(\theta + \phi)^2. \quad (2.16)$$

This represents different ways of fibering an  $S_1$  over another  $S_1$ . The fibration becomes trivial only when  $\alpha = \beta$ .

Consider however the case when  $N_2 = 0$  (i.e. one single stack of parallel branes), our solution reduces to the solution of single set of parallel 7-branes by transforming back to the  $z$  and  $w$  coordinates, and rescaling  $z = (\tilde{z}/(N_1 \sqrt{\beta}))$ , and similarly  $w = (\tilde{w}/(N_1 \sqrt{\alpha}))$ . The solution of parallel D7-branes is as expected independent of the choice of  $\alpha$ .

### 2.1.1 Different choices of coordinates

Our ansatz for solving the Einstein equations requires us to define a new set of coordinates  $x, y$  such that all non-trivial coordinate dependence is in  $y$ . This suggests that while the choice of  $y$  is unique up to an arbitrary rescaling, we have far more freedom in choosing  $x$ . A more general choice of  $x$  would be

$$x = A \ln z + B \ln w, \tag{2.17}$$

where  $AN_2 - BN_1 \neq 0$  so that the Jacobian is non-singular. This change in the choice of coordinates is inequivalent to a mere coordinate transformation on the resultant metric and are genuinely different solutions. We therefore want to know what sort of geometry is obtained, even though we started off with the same ansatz for  $\tau$ . Solving the Einstein equation as before with this different choice of coordinate, gives

$$\begin{aligned} K_{x\bar{x}} &= \alpha \exp\left(\frac{N_2 - N_1}{N_2A - N_1B}(x + \bar{x})\right) \\ K_{y\bar{y}} &= \beta \exp\left(\frac{B - A}{N_2A - N_1B}(y + \bar{y})\right) |\eta^2 \exp\left(\frac{y}{12}\right)|^2 \tau_2, \end{aligned} \tag{2.18}$$

where in this case  $\alpha\beta = 1/(AN_2 - BN_1)^2$ . For  $N_1 = N_2 = N$  we see that the dependence on  $A$  and  $B$  drops out from the exponential, though this is not true for  $N_1 \neq N_2$ . This again implies that the solutions for  $N_1 \neq N_2$  are qualitatively different from those when  $N_1 = N_2$ . Another issue however is that even for  $N_1 = N_2$ , this different choice of coordinate still has a non-trivial effect on the boundary conditions on the angles. This is most clearly exhibited in the case where  $N_1 = N_2 = 12$ . In that case both  $K_{x\bar{x}}$  and  $K_{y\bar{y}}$  become flat and naively we would get two cylinders. However similar considerations as in eq. (2.16) in the previous section implies that the compact part of the line element has non-trivial dependence on  $A$  and  $B$  i.e.

$$ds_{\theta,\phi} \sim \beta d(\theta + \phi)^2 + \alpha d(A\theta + B\phi)^2. \tag{2.19}$$

In effect we again have two entangled cylinders.

## 2.2 More general solutions

### 2.2.1 Separated branes

We would like to generalise the solutions we have obtained so far to the case where instead of coincident branes we have branes separated in each stack. The obvious ansatz for  $\tau$  would be

$$j(\tau) = \prod_{i,j} \frac{1}{(z - a_i)(w - b_j)}. \tag{2.20}$$

However, with this ansatz the coordinate transformation we have been using would produce a Jacobian that is no longer expressible as a product of  $f(x)g(y)$  for some holomorphic functions  $f, g$  and so we could no longer obtain block diagonal metric in  $x$  and  $y$ . There is

no obvious way to get around with this problem. On the other hand, it is possible to solve the Einstein equations when only one stack of branes separates. In that case

$$j(\tau) = \prod_i^{N_2} \frac{1}{z^{N_1}(w - b_i)} \tag{2.21}$$

and the choice

$$\begin{aligned} y &= -[N_1 \ln z + N_2 \ln w] \\ x &= B \ln w. \end{aligned} \tag{2.22}$$

gives

$$\begin{aligned} K_{x\bar{x}} &= \alpha \exp\left(\frac{x + \bar{x}}{B}\right) \prod_i^{N_2} \left| \exp\left(\frac{x}{B} - b_i\right) \right|^{-\frac{2}{N_1}} \\ K_{y\bar{y}} &= \beta \exp\left(-\frac{y + \bar{y}}{N_1}\right) |\eta|^2 \exp\left(\frac{y}{12}\right)^2 \tau_2, \end{aligned} \tag{2.23}$$

where  $\alpha\beta = 1/(N_1 B)^2$  here. At distances much greater than the separation of the branes, the solution reduces to that of coincident branes, as should be expected.

### 2.2.2 Branes intersecting at angles

To describe N branes intersecting at angles, the ansatz for  $\tau$  would be

$$j(\tau) = \prod_i^N \frac{1}{z - a_i w}. \tag{2.24}$$

The relevant choice of  $(x, y)$  coordinates is now

$$\begin{aligned} y &= -[N \ln w + \sum_i (\ln(\frac{z}{w} - a_i))] \\ x &= B \ln \frac{z}{w}. \end{aligned} \tag{2.25}$$

Fortunately the Jacobian of this coordinate change is  $|BN/zw|^2$  and we can easily repeat the same procedure for solving the Einstein equations to get

$$\begin{aligned} K_{x\bar{x}} &= \alpha \exp\left(\frac{x + \bar{x}}{B}\right) \prod_i^N \left| \exp\left(\frac{x}{B} - b_i\right) \right|^{-\frac{4}{N}} \\ K_{y\bar{y}} &= \beta \exp\left(-2\frac{y + \bar{y}}{N}\right) |\eta|^2 \exp\left(\frac{y}{12}\right)^2 \tau_2, \end{aligned} \tag{2.26}$$

and  $\alpha\beta = 1/(BN)^2$  here. The solution is locally identical to the above solution for separated branes in one of the two stacks with  $N_1/2 = N$ , except they satisfy different boundary conditions, meaning that the compact direction corresponding to the angles are entangled in a different manner in the two solutions.

### 2.2.3 Solutions not block-diagonal in $x, y$

So far we have only solved the Einstein equations assuming that the metric is block diagonal in  $x, y$  so that we can easily construct the corresponding Kahler potential. However, there is actually another class of solutions whose Kahler potentials also take a simple form. Consider a Kahler potential of the form

$$K = g(y, \bar{y}) + h(x, \bar{x})k(y, \bar{y}). \tag{2.27}$$

The determinant of the resultant metric is

$$(g_{,y\bar{y}} + hk_{,y\bar{y}})h_{,x\bar{x}}k - |h_{,x}k_{,\bar{y}}|^2. \tag{2.28}$$

If we could arrange  $K$  such that  $hk_{,y\bar{y}}h_{,x\bar{x}} - |h_{,x}k_{,\bar{y}}|^2 = 0$  then the determinant would become  $g_{,y\bar{y}}h_{,x\bar{x}}k$  which is again a simple product of a function in  $y$  and  $x$  respectively and should solve the Einstein equations. There are two simple distinct cases where this could be achieved. We could have

$$h(x, \bar{x})k(y, \bar{y}) = |m(x)n(y)|^2, \tag{2.29}$$

for any holomorphic function  $m, n$ .

A second choice would be to put

$$\begin{aligned} h(x, \bar{x}) &= [m(x) \pm m(\bar{x})]^A, \\ k(y, \bar{y}) &= [n(y) \pm n(\bar{y})]^B, \end{aligned} \tag{2.30}$$

such that  $A + B = 1$ .

To prevent extra factors of  $x$  appearing in the determinant of the metric other than the terms already appearing in the Einstein equation (2.11),  $h$  has to be quadratic in  $x$  in the case  $N_1 = N_2$ . For  $N_1 \neq N_2$  the component  $K_{x\bar{x}}$  has an exponential factor and that seems to dictate that we should choose the first set of solutions (2.29). As we shall see later this new set of solutions is perhaps necessary in helping us to obtain a metric of the conifold close to the intersection region of two orthogonal D7-branes.

## 3. A few observations about the solution

We would now like to explore further some of the implications of our solution. We will concentrate on the case where  $N_1 = N_2$  with the standard choice of  $x = \ln z - \ln w$  and  $y = -(\ln z + \ln w)$ , which was the basic choice made in section (2.1).

### 3.1 Angle deficit

Superficially the solution in section (2.1) looks like a single D7-brane in the  $(x, y)$  coordinate system. However, the transformation (2.9) implies that the  $x, y$  coordinates have non trivial boundary conditions and do not span the complex planes, since  $z, w$  do. Using the definition (2.9) and considering the simple case where  $N_1 = N_2 = N$ , we find

$$y_2 = -(\theta + \phi), \quad x_2 = (\theta - \phi), \tag{3.1}$$



where we write

$$y = y_1 + iy_2, \quad x = x_1 + ix_2$$

and

$$z = |z|e^{i\theta}, \quad w = |w|e^{i\phi}, \tag{3.2}$$

and for simplicity we have scaled away the factor of  $N$ .

Since  $\theta$  and  $\phi$  are periodic with period  $2\pi$ , the fundamental domain in the  $(x_2, y_2)$  plane can be chosen to be a parallelogram with base  $4\pi$  and height  $2\pi$ . Since the periodicities of  $x_2$  and  $y_2$  are linked we have to specify the path we are traversing before we can discuss the angle deficit. Furthermore, there is no meaning to the number of times the  $y$ -plane wraps around the fundamental region in the calculation of the energy of the system. We also see that for general values of  $N_1$  and  $N_2$  the size and shape of the fundamental region are different from the case where  $N_1 = N_2$ , and this suggests that the solutions for the former case are qualitatively different from the latter. In order to account for the correct periodicities we shall use the coordinates  $\theta, \phi$  instead of  $x_2, y_2$ . Although this complicates the expression for  $\tau$ , the metric nevertheless simplifies in the near horizon limit (i.e. when  $z, w \rightarrow 0$ ) This will be applied in the calculation of the energy of the system in a later subsection.

Returning to the discussion of angle deficit, the geometry of an individual D7-brane is associated with a deficit angle of  $\pi/6$  asymptotically far from its core. However, as described in [3] for stringy cosmic string, the solution is smooth close to the core. The geometry with the intersecting D7-branes is more subtle since each D7-brane affects the world-volume of the other. We would expect the angle deficit given by our solution of intersecting D7-branes to reduce to that of parallel D7-branes by taking first the limit to a region far away from one of the stacks of branes and then moving asymptotically far away from the second stack of branes. The angle deficit is computed by moving around the second set of branes in the plane transverse to it (say the  $z$ -plane) while keeping  $w$  fixed. In such a case we would be traversing simultaneously the  $x$  and  $y$  plane. In this limit

$$K_{y\bar{y}} = \beta \exp\left(-\frac{N_2 + N_1}{N_1^2 + N_2^2}(y + \bar{y})\right) |F|^2 \tau_2 \sim \exp\left(\left(\frac{1}{12} - \frac{N_2 + N_1}{N_1^2 + N_2^2}\right)(y + \bar{y})\right). \tag{3.3}$$

By a further coordinate transformation given by

$$\begin{aligned} \alpha \exp\left(\frac{N_2 - N_1}{N_1^2 + N_2^2}(x + \bar{x})\right) dx d\bar{x} &= \alpha \exp(P(x + \bar{x})) dx d\bar{x} = \alpha \frac{d\sigma d\bar{\sigma}}{P^2} \\ \beta \exp\left(\left(\frac{1}{12} - \frac{N_2 + N_1}{N_1^2 + N_2^2}\right)(y + \bar{y})\right) dy d\bar{y} &= \beta \exp(Q(y + \bar{y})) dy d\bar{y} = \beta \frac{d\eta d\bar{\eta}}{Q^2} \end{aligned} \tag{3.4}$$

the angle deficits in the  $\eta$  plane is exactly what one would expect for parallel coincident D7-branes. For  $N_1 = N_2$  the  $x$  plane becomes a cylinder without the need for a further coordinate transformation. In general the effect of the orthogonal set of branes cannot be removed by moving away from them.

Now consider a path that is simultaneously far from both stacks of branes and encircles the intersection region. Such a path keeps  $x$  fixed and traverses a circle in the  $y$ -plane.

By considering again the simple case  $N_1 = N_2 = N$ , the path traverses through an angle of  $2\pi$  in both the  $z$ -plane and the  $w$ -plane, resulting in an angle deficit in the  $\eta$  plane of  $2N\pi/6$ , which is the combined effect of  $2N$  branes. In particular, it should be noted that when  $N_1 = N_2 = 12$  both the  $x$  and  $y$  plane become cylinders and each plane has an angle deficit of  $2\pi$ , independent of the path we take. As remarked these cylinders are non-trivially fibred and only disentangle when  $\alpha = \beta$ .

### 3.2 Monodromies

The discussion in the previous section carries over to the analysis of monodromies. Consider a closed path traversing the  $z$  plane with  $w$  fixed. This translates into the  $x, y$  coordinates as a closed curve about  $x$  and  $y$  simultaneously. When we are sufficiently close to the stack of  $N_1$  branes transverse to the  $z$ -plane then  $C_0 = \tau_1 \sim \frac{N_1}{2\pi}\theta_z$  and we get exactly the monodromy we would expect around  $N_1$  branes.

Conversely if we consider moving in a closed curve in  $y$  keeping  $x$  fixed we would be moving in a curve which simultaneously moves around the  $z$  and  $w$  plane. If we integrate the RR 0-form field strength about a closed curve in the  $y$  plane in the region where  $|y|$  goes to zero, we get

$$\oint \partial_{y_2} C_0 dy_2 \sim \int_0^{2\pi(N_1+N_2)} \frac{-1}{2\pi} dy_2 = -(N_1 + N_2), \tag{3.5}$$

since  $C_0 = \tau_1 \sim \frac{-1}{2\pi}y_2$  as  $|y|$  approaches zero. This suggests a non-trivial combination of the monodromies when we move around both stacks of branes along a curve corresponding to constant  $x$ . The resultant monodromy is that of a stack of  $N_1 + N_2$  D7-branes transverse to  $y$ .

Note that while  $C_0$  depends only on one variable  $y$ , there are in fact more than one homotopically distinct classes of loops. This is because when we try to smoothly deform a set of loops that circles one set of the branes to loops circling the other set, we inevitably hit singularities corresponding to the brane core (i.e. at  $z = 0$  and  $w = 0$ ). This is analogous to the case with  $N$  parallel but non-coincident 7-branes. In that case everything depends only on one complex coordinate  $z$  but there are singularities at  $N$  different points in the complex plane and there are  $N$  homotopically inequivalent classes of loops.

### 3.3 Issues concerning enhanced supersymmetry

For two D7-branes intersecting over six spacetime dimensions, we should expect only a quarter of supersymmetry to be preserved. Yet the metric in the  $x$  plane can always be made flat suggesting that the amount of supersymmetry is doubled everywhere, contradicting the prediction of string perturbation theory. This is, however, an illusion. This is related to the non-trivial boundary conditions in the  $(x, y)$  coordinate system as discussed earlier. To make this explicit, consider the Killing spinor equation<sup>1</sup>

$$\delta\psi_a = D_a\epsilon = \left(\partial_a + \frac{1}{4}w_a^{A\bar{B}}\Gamma_{[A\bar{B}]} + \frac{1}{4}w_a^{\bar{A}B}\Gamma_{[\bar{A}B]} + \frac{i}{4}\frac{\partial_a\tau_1}{\tau_2}\right)\epsilon = 0 \tag{3.6}$$

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<sup>1</sup>We are adopting the notations in [5].

where  $w_a^{A\bar{B}}$  are the spin connections (whose non-trivial components are along the relatively transverse directions only).  $A, \bar{B} \in \{1, 2\}$  are complex tangent space indexes while  $a, \bar{b}$  are complex spacetime indexes corresponding to  $z$  and  $w$ . We need the projections

$$\Gamma^{\bar{A}}\epsilon = 0, \Gamma^A\epsilon^* = 0. \tag{3.7}$$

This ansatz thus explicitly preserves only a quarter supersymmetry. Yet, the trivial dependence on the  $x$  coordinate implies that its corresponding spin-connection should vanish, rendering the projections along  $x$  unnecessary. As a result supersymmetry is seemingly enhanced. We will show that the non-trivial boundary conditions inherited from the coordinate transformation ensures that supersymmetry is enhanced only locally, but not globally.

Using properties of Kahler manifolds,

$$\sum_A w_a^A \bar{e}_{\bar{b}}^{\bar{B}} = -\partial_a \ln \det \bar{e}_{\bar{b}}^{\bar{B}} \tag{3.8}$$

where  $e_a^B \bar{e}_{\bar{b}}^{\bar{B}} = K_{a\bar{b}}$ , the killing equation reduces to

$$\partial_a \ln \det \bar{e}_{\bar{b}}^{\bar{B}} = \frac{1}{2} \partial_a \ln \tau_2. \tag{3.9}$$

whose solution is

$$\det K_{z\bar{w}} = |F(z, w)|^2 \tau_2, \tag{3.10}$$

which is automatically a solution to the Einstein equation (2.2). After the coordinate transformation specified in (2.9) this becomes

$$\det K_{x\bar{y}} = |zw|^2 |F(x, y)|^2 \tau_2. \tag{3.11}$$

The factor of  $|zw|^2$  originates from the Jacobian of the transformation. In the presence of this term, eq (3.11) solves eq (3.9) everywhere except at the origin. This can be made explicit by substituting eq. (3.11) into (3.9) which gives

$$\partial_a \ln \det \bar{e}_{\bar{b}}^{\bar{B}} = \frac{1}{2} \partial_a \ln \tau_2 + \partial_a \ln \bar{z}\bar{w}. \tag{3.12}$$

This extra term is zero everywhere except at  $z = 0$  or  $w = 0$ . This is analogous to a conical space whose metric  $ds^2 = |z^n|^2 dz d\bar{z}$  can be made flat locally by the transformation  $z^n dz = d\eta$ . The coordinate transformation is singular at the origin and the spin connection is proportional to  $\partial_z \ln \bar{z}$  as in our solution. As is well known a conical space in general breaks all supersymmetry. For our solution, it takes us back to a quarter supersymmetric state. In other words, we should evaluate the spin connection in the  $z, w$  coordinates, which makes it explicit that supersymmetry is enhanced everywhere except at  $z, w = 0$ .

### 3.4 Energy of the system - the ADM mass

For a single cosmic string, the integral

$$E = \frac{1}{2} \int d^2z \sqrt{-g} R \tag{3.13}$$

is related to the energy of the system. In the  $x, y$  coordinates the space is asymptotically flat and we can define the ADM mass as usual. The above relation is obtained by using the  $R_{00}$  component in the Einstein equation which relates it to the energy momentum tensor. The result of the integral can be found by changing coordinates from  $(x_1, x_2, y_1, y_2)$  to  $(x_1, y_1, \theta, \phi)$  to avoid the awkward identification of the imaginary parts of  $x$  and  $y$ . Since the case where  $N_1 \neq N_2$  is qualitatively different from the case  $N_1 = N_2 = N$ , we consider here only the latter situation.

$$E = \frac{1}{2} \int \sqrt{-g} R d^2 x d^2 y = \frac{1}{2} \int \sqrt{-g} \partial_a \bar{\partial}_b [\ln(\det g_{m\bar{n}})] g^{a\bar{b}} d^2 x d^2 y. \quad (3.14)$$

Integrating by parts, we have

$$\frac{1}{2} \int d^2 x d^2 y \partial_a (\sqrt{-g} \bar{\partial}_b \ln H g^{a\bar{b}}) - \bar{\partial}_b \ln H \partial_a (\sqrt{-g} g^{a\bar{b}}), \quad (3.15)$$

where  $H = \det g_{m\bar{n}} = \tau_2 |\eta^2(\tau) e^{\frac{N}{12}(y+\bar{y})-1}|^2$ .

The second term vanishes. Since for a tensor of weight one the covariant derivative acts by

$$\nabla_a \tilde{T}^{a\bar{b}} = \partial_a \tilde{T}^{a\bar{b}} + \Gamma_{am}^a \tilde{T}^{m\bar{b}} + \Gamma_{am}^{\bar{b}} \tilde{T}^{am} - \Gamma_{ma}^m \tilde{T}^{a\bar{b}} \quad (3.16)$$

and  $\Gamma_{am}^{\bar{b}}$  is zero for a Kahler metric. Using this relation we can write eq. (3.15) as

$$E = \frac{1}{4} \int \partial_x [(\sqrt{-g} g^{x\bar{x}}) \bar{\partial}_x \ln H] + \bar{\partial}_x [(\sqrt{-g} g^{x\bar{x}}) \partial_x \ln H] + \partial_y [(\sqrt{-g} g^{y\bar{y}}) \bar{\partial}_y \ln H] + \bar{\partial}_y [(\sqrt{-g} g^{y\bar{y}}) \partial_y \ln H]. \quad (3.17)$$

Then we express the integrand in terms of the real parts of  $x$  and  $y$  i.e.  $x_1$  and  $y_1$ . For the imaginary parts we revert to the original angular variables of  $z$  and  $w$ . i.e.

$$x_2 = \theta - \phi, \quad (3.18)$$

$$y_2 = -(\theta + \phi). \quad (3.19)$$

The integral is thus over  $\mathfrak{R}^2 \times T^2$ . Concentrate on the first 2 terms of the integrand involving derivatives with respect to  $x_1$  and  $y_1$ , the integrand is

$$\frac{1}{8} \int d\theta_z d\phi_w dy_1 dx_1 \{ \partial_{x_1} [(\sqrt{-g} g^{x\bar{x}}) \partial_{x_1} \ln H] + \partial_{y_1} [(\sqrt{-g} g^{y\bar{y}}) \partial_{y_1} \ln H] \}. \quad (3.20)$$

The first term is zero since  $H$  is independent of  $x$  and we are left with the second term. Being a total derivative it reduces to

$$\int dx_1 d\theta_z d\phi_w [\sqrt{-g} g^{y\bar{y}} \partial_{y_1} \ln H]_{-\Lambda}^{\Lambda}, \quad (3.21)$$

where  $\Lambda \rightarrow \infty$ . In this limit where  $y_1 \rightarrow \infty$ ,

$$(\sqrt{-g} g^{y\bar{y}}) \partial_{y_1} \ln H \rightarrow 2\alpha \left( \frac{N}{12} - 1 \right). \quad (3.22)$$

In the other limit where  $y_1 \rightarrow -\infty$ ,

$$(\sqrt{-g}g^{y\bar{y}})\partial_{y_1} \ln H \rightarrow \alpha\partial_{y_1}(-2y_1 + \ln \frac{y_1}{2\pi}) \rightarrow -2\alpha. \tag{3.23}$$

As a result (3.21) becomes

$$\int dx_1 (4\pi^2)\alpha(\frac{N}{6} - 2 + 2). \tag{3.24}$$

This diverges linearly in  $x_1$  as expected due to the extra dimension growing out of the intersection region. The extra factor of  $4\pi^2$  originates from the two  $SO(2)$  symmetries. We should rescale  $x_1$  by  $\sqrt{\alpha}$  so that it is identified with the extra isometry direction. A factor of  $\sqrt{\alpha}$  is left in the integral. Thus  $\alpha$  parametrises a family of different one-quarter BPS geometries. The fact that the energy of a BPS solution depends on one or more parameters (moduli) is familiar, for example, for the monopole in Yang-Mills-Higgs theory.

### 3.5 Some final remarks

As we shall see in the next section,  $j(\tau)$  can be equated to more general polynomials in  $z$  and  $w$ .

$$j(\tau) = \frac{P(z, w)}{Q(z, w)} \tag{3.25}$$

As long as the polynomials are functions of  $z^{N_1}w^{N_2}$  or one of the above generalised forms, we can solve them in exactly the same manner, except that the form of  $F$  should be modified accordingly to

$$F = \eta(\tau)^2 Q(z, w)^{-\frac{1}{12}}. \tag{3.26}$$

As mentioned earlier the solution has not been generalised to the case where both stacks of branes are separated. This is because if we alter our ansatz to accommodate this situation by eq. (2.20) there are no obvious new variables that would allow separation of variables in the equations of motion.

Solutions of intersecting D7 and  $(p, q)$  7-branes have been obtained in [6], in which the complex modulus  $\tau$  is arranged to have dependence only on one of the complex coordinates,  $z$ . Using the relation of compactified F-theory with type IIB on orientifolds [7, 8], the solution describes two orthogonal stacks of D7-branes and orientifold planes. One stack consisting of sixteen D7-branes coinciding with four orientifold planes is transverse to the  $w$ -plane. The other stack consisting of another sixteen D7-branes and orientifold planes separated, is transverse to the  $z$ -plane. The form of these solutions is very similar to those explored here except they are block diagonal in  $z, w$ . Yet our solutions have completely different interpretations. We have different monodromies that signify intersecting D7-branes separated from any orientifold planes — while not apparent in the  $x, y$  coordinates this could be read off from the  $z, w$  coordinates. The fact that we could distinguish these solutions is encoded in the different boundary conditions i.e. the compact directions in  $x, y$  satisfy different periodicities.

## 4. Intersecting D7-branes and conifolds

### 4.1 Conifolds and intersecting branes

It was first observed in [4] that a set of intersecting NS5 branes over 3+1 dimensions in type IIB is the T-dual of type IIA compactified on a Calabi-Yau space. When the branes are arranged to intersect orthogonally and with no extra fields switched on, the T-dual (along one of the totally transverse directions) geometry is a conifold at the intersection. The singularities that occur in the dual geometry provide information of the gauge symmetry the intersecting branes possess. There is an ADE classification of singularities that could be described by algebraic varieties (see for example [9]) which can be matched with the Dynkin diagrams of groups. Therefore we can read off the kind of enhanced gauge symmetry in the intersecting brane theory from the geometric singularities that are present. The physical interpretation of the symmetry enhancement is that the presence of a singularity implies the collapse of some cycles upon which branes could wrap. As a result the branes become massless and we obtain extra massless states supplying the extra vector multiplets required for the symmetry.

The intersecting D7-branes are intimately related to intersecting NS5 branes. Recall that the D7-brane solution is an adaptation of the stringy cosmic string solution. Now suppose we go back to the original context where the stringy cosmic string solution was considered, i.e. type IIB compactified on a  $T^2 \times M$  for some four dimensional compact manifold  $M$  to four dimensions. Starting with the cosmic string solution with only a non-trivial complex modulus  $\tau$  but trivial Kahler modulus  $\rho = B + i\sqrt{G_{T^2}}$ , where  $B$  is the NS 2-form and  $G_{T^2}$  is the determinant of the metric on the torus, we can perform a T-duality along one of the cycles on the torus. This would exchange  $\rho$  and  $\tau$  and the monodromies in  $\tau$  would be transferred to  $\rho$ . As a result instead of an axion magnetic charge we obtained magnetic charge of the NS 2-form potential  $B$ . The magnetic charge is nothing other than the NS 5 brane in type IIA (or we could equally well have started with cosmic strings in type IIA). In the case of  $N$  parallel cosmic strings the T-dual 5 brane picture corresponds to  $N$  parallel 5 branes transverse to the complex  $z$  plane and the torus  $T^2$ . The manner in which T-duality relates geometry of type IIB (or IIA) with the NS5-brane configuration in type IIA (or IIB) was studied in [4]. The creation of singularities by varying the complex parameters of the geometry that leads to symmetry enhancement can now be understood from the T-dual point of view as NS5 branes coinciding.

It was then shown in [4] that type IIB(A) on  $T^2 \times T^2$  gives an analogous stringy cosmic string solution except that now there are two tori moduli  $\tau$  and  $\tau'$  fibred on the complex  $z$  plane and each can be described by a separate Weierstrass equation. If the two tori become singular at the same point in the  $z$  plane (say  $z = 0$ ), close to the singularity these equations are reduced to the form

$$\begin{aligned} \zeta_1 \zeta_2 &= z \\ \eta_1 \eta_2 &= z, \end{aligned} \tag{4.1}$$

where  $\zeta_i, \eta_j$  are all complex coordinates that parametrise the two tori respectively. Com-

binning the two equations we obtain

$$\zeta_1 \zeta_2 - \eta_1 \eta_2 = 0 \tag{4.2}$$

which is the conifold equation. The corresponding NS5 brane configuration is obtained by T-duality along a cycle of each torus as before, but this time results in intersecting NS5 branes over 3+1 dimensions. Each 5 brane is transverse to one of the tori. That intersecting singularities should lead to conifolds is quite a general phenomenon. In this particular case it is the collision of two  $A_0$  nodes that produces the conifold.<sup>2</sup>

Since intersecting D7-branes are basically the cosmic strings considered above the resultant geometry should carry intersecting singularities. Therefore it is very likely that we should be able to see a conifold from the F-theory picture, where there are enough dimensions to visualise the singularity.

#### 4.2 Intersecting D7-branes and F-theory

The conifold is a Calabi-Yau three-fold. In order to see the conifold that is allegedly lying at the intersection of two D7-branes, we need two extra dimensions in addition to the four relatively transverse dimensions. Therefore a discussion of the conifold only makes sense when we can establish the precise relation between the ten-dimensional type IIB with F-theory compactified on a Calabi-Yau. The compactification of F-theory on elliptically Calabi-Yau three-folds was considered in [10]. There are many different classes of such Calabi-Yau three-folds. In the simplest case the base space is the minimal ruled surface  $\mathbb{F}_n$  i.e. all possibilities where a 2-sphere is fibred on another 2-sphere. The total space can again be described by a Weierstrass equation as in (A.12) except that now  $f$  and  $g$  are functions of two complex variables  $z, w$ . In analogy to the case with parallel 7 branes  $f$  is a polynomial of degree 8 w.r.t to  $z$  and  $w$  separately, and similarly for  $g$ , which is now a degree 12 polynomial in  $z$  and  $w$ . In general they can be written as

$$\begin{aligned} \alpha^2 &= \beta^3 + f(z, w)\beta + g(z, w) \\ f(z, w) &= \sum_{i,j}^8 a_{ij} z^i w^j \\ g(z, w) &= \sum_{i,j}^{12} b_{ij} z^i w^j \end{aligned} \tag{4.3}$$

This suggests that the type IIB interpretation should contain 24 7-branes intersecting another set of 24 7-branes orthogonally over 5+1 dimensions, which should give a compact space of  $S^2$  fibred over  $S^2$ . A related observation is that in the supergravity solution we presented for  $N_1 = N_2 = 12$ , the space becomes a product of two cylinders. Therefore, as in [3], when there are 24 branes on each stack we would glue the cylinders together to get two  $S^2$ , consistent with the F-theory picture. The zero locus of the discriminant of the Weierstrass should give the position of the branes. In general the branes wrap around curves

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<sup>2</sup>It should be noted however that despite several attempts (e.g. [11, 12]), so far the conifold metric has not been satisfyingly obtained from intersecting 5-brane supergravity solutions.

in the  $z$  and  $w$  plane. The total number of complex moduli modulo  $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$  and rescaling of  $f$  and  $g$ , is  $9 \times 9 + 13 \times 13 - 3 - 3 - 1 = 243$ . This corresponds to families of (3,243) Calabi-Yau manifolds. In general, such F-theories should correspond to orientifold type IIB theories [13]. We shall briefly review the arguments. To see the orientifold planes and D7-branes, we first scale into a particular region where  $f$  and  $g$  can be treated as polynomials of degree (2,2) and (3,3), respectively, and where they take special forms

$$\begin{aligned} f(z, w) &= C\eta(z, w) - 2h(z, w), \\ g(z, w) &= h(z, w)(C\eta(z, w) - 2h(z, w)) \end{aligned} \tag{4.4}$$

for some polynomials  $\eta$  and  $h$ , and  $C$  is a constant that is tuned to be small such that from

$$j(\tau) = \frac{4(24)^3(C\eta - 3h^2)^3}{C^2\eta^2(4C\eta - 9h^2)} \tag{4.5}$$

we can arrange  $\tau \sim i\infty$  (i.e. the weak coupling limit in which type IIB description makes sense). Then the positions of D7-branes correspond to zeroes of the determinant  $\Delta = C^2\eta^2(4C\eta - 9h^2)$ , which gives some colliding  $A_1$  singularities (depending on the precise form of  $\eta$ . e.g. for  $\eta = z^2w^2$  we would have two colliding  $A_1$  singularities). Further, examining the monodromies of  $\tau$  around  $\eta$  gives an  $SL(2, \mathbb{Z})$  transformation of

$$T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2, \tag{4.6}$$

and so is consistent with the interpretation as a pair of coincident (and perhaps curved in  $z, w$  planes) D7-branes. The other zeroes of  $\Delta$  occur at

$$h(z, w) = \pm \frac{2}{3} \sqrt{C\eta(z, w)}. \tag{4.7}$$

Again in the limit of small  $C$  the zero locus is approximately  $h = 0$  and we can look into the monodromies of  $\tau$  when we move about  $h = 0$ . This would give the composition of the monodromies from each of the two curves obtained in (4.7) and using (4.5) gives an  $SL(2, \mathbb{Z})$  transformation,  $\pm T^{-4}$ . Therefore it is consistent to interpret these curves (4.7) as two coincident orientifold planes in weakly coupled type IIB. While we started off with some products of  $SU(2)$  gauge symmetries (or  $A_1$  singularities) we can break these symmetries to obtain more general brane configurations. This can be done by allowing  $g$  to be deformed to

$$g = h(C\eta - 2h^2) + C^2\chi, \tag{4.8}$$

for some arbitrary polynomial  $\chi$  subjected to the same restriction as  $g$ . Again taking  $C \rightarrow 0$  and  $h, \eta, \chi$  fixed we go to the weak coupling limit, and we find that the discriminant becomes

$$\Delta = -9C^2h^2(\eta^2 + 12h\chi). \tag{4.9}$$

It is clear that the original coincident D7-branes given by  $\eta = 0$  are now *separated*. In this manner the breaking of the  $SU(2)$  symmetry by deforming  $g$  to obtain the most arbitrary



configurations is equivalent, from the type IIB picture, to the Higgs mechanism where branes move apart. In order to reconstruct the complete  $f$  and  $g$  that describe all the branes we would need to patch different regions which locally admit such a description. One example is the correspondence between F-theory on  $\mathbb{F}_\times$  (where the base space is simply the direct product of two 2-sphere) and the T-dual of the Gimon-Polchinski model [8, 14, 15]. In this correspondence it was observed that the number of constraints needed to obtain an  $(\text{SU}(2))^8 \times (\text{SU}(2))^8$  gauge symmetry exceeds the total number of complex moduli. Fortunately there are still families of solutions that could satisfy all the constraints but these are constrained to lie in a particular slice of the moduli space. As a result the total number of massless hypermultiplets is less than 243. But the number of extra neutral hypermultiplets that appear as  $\text{SU}(2)$  is broken matches the F-theory prediction as the constraints on  $f$  and  $g$  are lifted. Returning to the question of the search for the conifold, that arises from the orthogonal intersection of two D7-branes, we could analyse the geometry by scaling close to the intersection region while arranging the orientifold planes to be sufficiently far so that they do not interfere with the local geometry. This will be carried out in the next section.

### 4.3 Conifold in F-theory

Having now reviewed the relation between F-theory compactified on an elliptically fibred Calabi-Yau three-fold with type IIB orientifold theory, the various parameters of the Weierstrass equation can now be varied to build a conifold in F-theory which can be interpreted as orthogonally intersecting D7-branes in type IIB. For this to happen two  $A_0$  nodes corresponding to  $\text{U}(1) \times \text{U}(1)$  gauge symmetry must collide. From (A.12) it is straight forward to find the specific forms of  $f$  and  $g$  needed to construct these colliding  $A_0$  singularities,

$$\begin{aligned} f &= a_{00} + a_{11}zw + O(z^2, w^2) \\ g &= b_{00} + b_{11}zw + O(z^2, w^2). \end{aligned} \tag{4.10}$$

The discriminant is then

$$\Delta = 4f^3 + 27g^2 = (4a_{00}^3 + 27b_{00}^2) + zw[A + O((zw)^2)], \tag{4.11}$$

where  $A$  is just some combination of  $a_{11}$  and  $b_{11}$ . In order to describe colliding  $A_0$  singularities at  $z = w = 0$  we need to set

$$(4a_{00}^3 + 27b_{00}^2) = 0, \tag{4.12}$$

and

$$\begin{aligned} \alpha &= 0 \\ \beta &= \sqrt{-\frac{a + zw + O(z^2w^2)}{3}}. \end{aligned} \tag{4.13}$$

These are the points where (A.12) is satisfied and all its partial derivatives vanish. They are the singular points. Expanding (A.12) about these singular points up to second order, we get

$$\delta_\beta^2 - \delta_\alpha^2 + b_{00}zw = 0, \tag{4.14}$$

which is the conifold equation. To summarise, the conifold equation is built from two requirements that are necessary if it is to represent intersecting D7-branes. First, the zero locus of  $\Delta$ , which is interpreted as the position of the D7's, should represent two orthogonally intersecting  $A_0$  nodes. Second, the monodromies about these 7-branes should correspond to those of D7-branes.

The next obvious question to ask is then whether we could obtain the conifold metric from our metric for intersecting D7-branes by applying the standard procedure described in [3], which extends the lower dimensional metric to a higher dimensional one including the torus. This presents difficulties, as we will now see.

#### 4.4 Obstruction to obtaining the conifold metric from intersecting 7 branes solution

We would like to extend our solution of the ten dimensional Einstein equations to twelve dimensions in order to visualise the conifold.<sup>3</sup> It was argued in [3] that if the total space  $E$ , which is the elliptically fibred Calabi-Yau  $n$ -fold F-theory is compactified on, admits a nowhere vanishing holomorphic  $(n, 0)$  form  $W$  and that we can put a Kahler metric on  $E$  whose corresponding volume form approaches  $W \wedge \bar{W}$  asymptotically, then  $E$  admits a Ricci-flat Kahler metric for non-compact  $E$  [3]. Since we are scaling into the conifold it is reasonable to consider  $E$  to be non-compact. Following these lines the corresponding  $(3, 0)$  form  $W$  in our solution would be

$$W = \Delta^{-\frac{1}{12}} \eta^2(\tau) d\zeta dx dy, \tag{4.15}$$

where  $\zeta$  is a complex coordinate on the torus with periodicity

$$\zeta \sim \zeta + 1 \sim \zeta + \tau, \tag{4.16}$$

and  $\Delta \sim (zw[A + O((zw)^2)])$  is as defined in eq. (4.11).

The Kahler potential for this extra torus which has all the required properties, including invariance under  $SL(2, \mathbb{Z})$  transformation, is then given by

$$\Phi_{T^2} = -\frac{(\zeta - \bar{\zeta})^2}{2\tau_2}. \tag{4.17}$$

This part of the potential gives the metric on the F-theory torus,

$$ds_{T^2}^2 = \frac{1}{\tau_2} |d\zeta - \frac{\zeta - \bar{\zeta}}{2i\tau_2} \partial_y \tau dy|^2. \tag{4.18}$$

This metric is automatically Kahler flat since the 12-dimensional Ricci scalar would be

$$R = G^{a\bar{b}} \partial_a \partial_{\bar{b}} (\ln g_{10} - \ln \tau_2) = 0 \tag{4.19}$$

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<sup>3</sup>Certainly we should be very cautious with this search for metrics because the compact torus in F-theory is actually 1+1 dimensions and yet we are treating them as purely Euclidean.

where  $g_{10}$  is the determinant of the 10-dimensional Kahler metric and we immediately identify what appears in the bracket as the 10-dimensional Einstein equation. Now take the limit  $zw \rightarrow 0$  or  $y_1 \rightarrow \infty$ , then

$$\begin{aligned} \tau &\sim \frac{y}{2\pi i} \\ K_{y\bar{y}} &\sim \exp(-2y_1) \frac{y_1}{8\pi} \end{aligned} \tag{4.20}$$

The total Kahler potential in this limit is (keeping only leading order terms)

$$K \sim \frac{2\zeta_2^2}{y_1} + |x|^2 - \frac{e^{-2y_1} y_1}{8\pi}. \tag{4.21}$$

Note that the third term is exponentially suppressed compared to the first two terms. We wish to compare this Kahler potential with that of the conifold. The Kahler potential of a conifold is [16]

$$\Phi_{\text{conifold}} = \frac{3}{2} r^{\frac{4}{3}} \tag{4.22}$$

where

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 0 \\ |a|^2 + |b|^2 + |c|^2 + |d|^2 &= r^2. \end{aligned} \tag{4.23}$$

In order to compare the two Kahler potentials it would be more convenient to change coordinates to  $\zeta, x, y$ . The torus coordinate  $\zeta$  is related to the coordinates in the Weierstrass equation (up to scaling) (4.3) by

$$\zeta = \int_{\beta_0}^x \frac{d\beta}{\alpha}, \tag{4.24}$$

along some path  $C$  in the complex  $\beta$  plane. Since we are now restricting to a region very close to the conical singularity  $\delta_\alpha^2 = \delta_\beta^2 = zw = 0$  we can ignore  $\delta_\beta^3$  and so we obtain

$$\zeta = \ln(\delta_\alpha + \delta_\beta) - \frac{1}{2} \ln zw. \tag{4.25}$$

Substituting into eq. (4.3) we have

$$\begin{aligned} \delta_\alpha &= \sqrt{zw} \cosh \zeta, \\ \delta_\beta &= \sqrt{zw} \sinh \zeta, \\ r^2 &= |zw| \cos 2\zeta_2 + \frac{|z|^2 + |w|^2}{2} = e^{-y_1} [\cosh x_1 + \cos 2\zeta_2]. \end{aligned} \tag{4.26}$$

Now the Kahler potential as a function of  $r$  can be expressed in our coordinates and the resulting metric can be shown to be Ricci flat, which is very much in the same spirit as [17]. We first solve the conifold equation to obtain the local coordinates on the conifold and then impose an  $SO(n)$  symmetry by obtaining the expression of  $r$  such that the Kahler potential is solely a function of  $r$ . Finally we can compute the Ricci scalar and set it to zero. The Kahler potential is then obtained from this differential equation. Comparing (4.26) with

the D7-brane Kahler potential (4.21) we are impressed that both depend on the same coordinates  $y_1, \zeta_2, x_1$  but (4.21) depends also on  $x_2$ . Considering the discussion in section 2.2.3 a possible modification to our Kahler potential could be

$$K_{\text{modified}} = \frac{2\zeta_2^2}{y_1} + \frac{x_1^2}{y_1} - \frac{e^{-2y_1}y_1}{8\pi}. \quad (4.27)$$

Ignoring also the third term since it is exponentially suppressed, we have

$$K_{\text{modified}} \sim \frac{2\zeta_2^2}{y_1} + \frac{x_1^2}{y_1}, \quad (4.28)$$

which now has a more similar structure to (4.26) and indeed is always positive and approaches zero as  $zw \rightarrow 0$ , and can therefore be identified with some radius of the conifold. However, it is intrinsically different from the conifold Kahler potential by its dependence on  $y_1$ , which cannot be remedied by a different choice of metric, in the base space. This is rooted in the form of the metric, which is block diagonal in the torus metric and the base space, breaking explicitly the required  $SO(4)$  symmetry of the conifold. This can be seen by a coordinate transformation

$$\zeta = t_1 + \tau t_2 \quad (4.29)$$

where  $t_i \sim t_i + \mathbb{Z}$  and  $t_i \in \mathbb{R}$ . The torus metric (4.18) then becomes

$$ds_{T^2}^2 = \frac{1}{\tau_2} |dt_1 + \tau dt_2|^2, \quad (4.30)$$

which is the standard torus metric with complex modulus  $\tau$  and unit metric determinant, consistent with the fact that the torus is unphysical since its Kahler modulus is not part of the physical spectrum and cannot be varied. The same problem also plagues the metric of K3 with  $A_1$  singularity in the parallel stringy cosmic string solution. In that case the  $SO(3)$  symmetry expected of a the conical singularity is again broken by the choice of the block diagonal ansatz.

This is similar to the observation in [18] where it is found that the T-dual to the stringy cosmic string solution does not give the exact NS5 brane metric because of the block diagonal ansatz of the metric in the torus and the transverse complex plane  $z$ . It was argued there that the correct solution should emerge if we go beyond the adiabatic approximation used in [3]. However, given that the extra dimensions are not physical and no momentum can propagate along those directions, it is questionable as to whether it is sensible to obtain a metric which preserves an  $SO(4)$  symmetry mixing these directions with those along the base space. Another observation is that in [19] the leading order metric of the moduli space of type IIA on a Calabi Yau three-fold near the conifold point is given by the stringy cosmic string solution. On the other hand we have seen here how the stringy cosmic string possibly describes a conifold directly, albeit with the need for some modifications. This seems to suggest some curious connection between a geometry and its moduli space.

Finally we would like to make a remark about deformed and resolved conifolds. From the derivation of (4.3) the deformed conifold equation

$$a^2 + b^2 + c^2 + d^2 = \epsilon \quad (4.31)$$

for small  $\epsilon$  can simply be obtained by replacing  $zw$  by  $zw - \epsilon$  everywhere. This would correspond to curved 7-branes, very much analogous to the NS5 brane configurations dual to a deformed conifold. However, learning from the NS5 story, we know that the deformed conifold is related to a resolved conifold by mirror symmetry via three T-dualities along three isometry directions [20], or by a conifold transition. So do our curved D7-branes producing the deformed conifold T-dualise to a resolved conifold? Also is there some D7-brane configuration which naturally produces a resolved conifold? The answer is probably no to both questions. This is because, as is well known, a resolved conifold is obtained from the conifold by varying the Kahler moduli, which are not physical modes in F-theory. The Kahler modulus of the torus is a priori fixed. From another perspective, in order to go over to the mirror manifold we have to perform three T-dualities and this must involve T-duality along one of the cycles of the fibred torus. Since no physical modes can propagate in the torus, it is not known whether T-duality along these directions makes any sense.

## 5. Discussion and conclusion

In this paper we have obtained several supergravity solutions that describe intersecting D7-branes. The solutions turn out to be very similar to parallel D7-brane solutions and the major difference lies in the non-trivial boundary conditions. This suggests that an enhancement of supersymmetry away from the singular point of the geometry where the branes reside is a quite generic phenomenon and it seems that the symmetry enhancement observed in intersecting D5-branes in [2] also occurs for intersecting D7's.

However, there is an important difference between intersecting D5's in [2] and D7 since the connection between the microscopic string description and supergravity is unclear for a system of D7-branes. It is impossible to take the large- $N$  limit since each D7-brane induces a deficit angle of  $\pi/6$  independent of the string coupling. It is therefore impossible to ignore the back-reaction of the branes on space-time. This is seen in the string description from the fact that closed string exchange cannot be ignored. At the very least, it is necessary to include the coupling of the RR-scalar,  $\tau_1$ , to the bulk Riemann tensor [1]. As a result a probe brane analysis using the DBI action could be quite subtle [21].

On the other hand, the geometry generated by intersecting D7-branes is of interest in its own right. For example the world-volume theory of a D3 brane embedded in the intersection domain of two stacks of D7-branes is  $N = 1$  QCD with flavors [22]. The supergravity solution of intersecting D7-branes discussed in this work should be a good starting point for exploring the gravity dual of  $N = 1$  QCD.

By exploring further the relations of 7-branes in type IIB with F-theory, it looks very likely that a conifold should emerge at the intersection of two orthogonal D7-branes from the 12-dimensional F-theory picture. However, we haven't been able to obtain the explicit conifold metric by extending our solution to twelve dimensions applying the procedure described in [3]. This is due to the breaking of the  $SO(4)$  symmetry of the conifold by the block diagonal ansatz of the metric which distinguishes the fibred torus from the base space. This breaking of symmetry is possibly intrinsic to F-theory since the extra dimensions are unphysical and it is hard to imagine how they could have mixed with the other physical

directions.<sup>4</sup> The deformed conifold can be obtained by turning the D7 branes into curved ones, very much analogous to the NS5 brane story. However, it is unlikely that there is a resolved conifold counterpart in the D7 picture. This is once again due to the unphysical nature of the extra dimensions, whose only remnant after BRST projection is the complex modulus  $\tau$ . The Kahler modulus is non-existent and it would not be possible to resolve the singularity by modifying the Kahler modulus. From the brane perspective, again comparing with the NS5 branes, D7-branes separated along the totally transverse directions would probably produce a resolved conifold. Yet these totally transverse directions are exactly those non-physical torus directions. The D7's are thus stuck together rendering a resolved conifold improbable.

Finally there are several clear aspects of these kinds of intersections that have not been explored. We would like to understand more about the proposed solution in the case where  $N_1 \neq N_2$  and the other families of solutions giving non-block-diagonal solutions.

We should understand how angle deficits can be evaluated in these situations and the way to generalise to non-coincident branes in both stacks or possibly other curved brane configurations. It would be interesting to see how the symmetry enhancement seen in the supergravity solution can be described in microscopic terms by string theory. It would also be interesting to understand if the conifold plays any role in these intersecting D7's and if there exists more general null projection in F-theory that could reproduce the 10 dimensional IIB metric from a twelve-dimensional starting point.

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## A. An overview of stringy cosmic strings, D7-branes and their relation to F-theory

### A.1 Stringy cosmic string and D7-branes

The supergravity solution of a cosmic string in four dimensions which induces non-trivial  $SL(2, \mathbb{Z})$  monodromies on the complex modulus  $\tau$  was considered in [3]. The complex modulus  $\tau$  originates from a torus on which a higher dimensional theory compactifies, therefore the theory should be invariant under  $SL(2, \mathbb{Z})$  transformation, where

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \tag{A.1}$$

and  $ad - bc = 1$ ,  $a, b, c, d$  are integers. In type IIB string theory it is believed that  $SL(2, \mathbb{Z})$  is an exact symmetry even in the quantum theory. The complexified string coupling combining the RR zero form and the string coupling, also usually known as  $\tau = C_0 + ie^{-\phi}$

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<sup>4</sup>In fact the search for such a metric might not make sense in the first place because these extra dimensions are not even Euclidean.

transforms as in (A.1) and the form of the IIB effective action looks exactly like the compactified supergravity theory mentioned above [3]. This suggests that the solution can be adopted for describing other extended objects of codimension 2, which are D7-branes in ten dimensions [23].

The bosonic part of type IIB supergravity is given by

$$\int \sqrt{g} \left( R + \frac{\partial_\mu \tau \partial_\nu \bar{\tau}}{2\tau_2^2} g^{\mu\nu} \right) \tag{A.2}$$

where only the dilaton and  $RR$  zero form are switched on.

The equations of motion obtained are:

i) The  $\tau$  equation:

$$\partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \tau) = 2\sqrt{-g} g^{\mu\nu} \frac{\partial_\mu \tau \partial_\nu \bar{\tau}}{\tau - \bar{\tau}}. \tag{A.3}$$

ii) Einstein equation:

$$R_{\mu\nu} = \frac{1}{4\tau_2^2} (\partial_\mu \tau \partial_\nu \bar{\tau} + \partial_\nu \tau \partial_\mu \bar{\tau}). \tag{A.4}$$

Note also that there is a composite U(1) connection

$$Q_\mu = i \frac{\partial_\mu \tau_1}{\tau_2} \tag{A.5}$$

that enters into covariant derivatives along with the spin connection. Suppose the D7-branes are aligned to be transverse to  $x^8$  and  $x^9$ . It will be more convenient to combine these two transverse dimensions into a complex coordinate  $z = x^8 + ix^9$  since the resultant geometry is Kahler. All fields depend only on  $z$ . Substituting the metric ansatz

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \Omega^2(z) dz d\bar{z}. \tag{A.6}$$

into (A.4) gives

$$2\partial\bar{\partial} \ln \Omega = \partial\bar{\partial} \ln \tau_2. \tag{A.7}$$

It turns out that any holomorphic  $\tau$  satisfies the equations and a convenient solution for the metric is  $\Omega = \tau_2$ . However this is not  $SL(2, \mathbb{Z})$  invariant and an acceptable solution is

$$j(\tau) = \frac{1}{z} \tag{A.8}$$

$$\Omega^2 = \tau_2 |\eta^2(\tau) z^{-1/12}|^2, \tag{A.9}$$

where  $j(\tau)$  maps the fundamental region in the  $\tau$  plane to a complex plane exactly once. In fact

$$j(\tau) = \frac{\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8}{\eta^{24}}. \tag{A.10}$$

The dedekind eta function is a modular function that compensates for the transformation in  $\tau_2$ . The factor of  $z^{-1/12}$  removes the singularity in the eta function as  $z$  approaches zero. It follows from (A.8) that  $\tau \rightarrow \infty$  as  $z \rightarrow 0$  and that  $\tau \rightarrow \tau + 1$  as  $z \rightarrow ze^{2\pi i}$ , signifying

the presence of a D7-brane at  $z = 0$ . As  $z \rightarrow \infty$ ,  $\tau$  approaches a constant value and the metric becomes

$$\Omega^2 = |z^{-1/12} dz|^2, \tag{A.11}$$

which implies an angle deficit of  $\pi/6$ . The solution generalises to describe more branes, by equating  $j(\tau)$  to more general holomorphic functions in  $z$ . Since each brane produces an angle deficit of  $\pi/6$ , 24 parallel branes have angle deficit of  $4\pi$ , in which case the transverse space is compactified to a sphere. In general the presence of angle deficits breaks all supersymmetries, as can be seen from the killing spinor equation, but the D7-brane solution is supersymmetric. This property follows from a cancellation of the effect of the spin connection by the U(1) connection.

### A.2 Relation to F-theory

As discussed above, in the presence of 24 branes the transverse space is compactified to a two sphere. In the stringy cosmic string context there is a torus whose modulus is  $\tau$  fibred over this sphere, where the total space is a K3 surface. However in the description of 7-branes  $\tau$  is the complexified string coupling and there are not any torus related to  $\tau$ . The analogy is so strong that it was suggested in [24] that type IIB with 24 7-branes is equivalent to twelve dimensional F-theory compactified on an elliptically fibred K3 [24]. The modulus of the fibre, the torus, is  $\tau$ . The two extra dimensions, however, are not dynamical and by a suitable projection operator no physical states propagate along these extra dimensions and the only remnant is  $\tau$ .<sup>5</sup> An elliptically fibred K3 can be conveniently described by a Weierstrass equation whose coefficients are functions of the base space, i.e.

$$y^2 = x^3 + f(z)x + g(z). \tag{A.12}$$

The zero locus of the discriminant of the equation

$$\Delta = 4f^3 + 27g^2 \tag{A.13}$$

gives the position of the 7 branes and the modulus of the torus is related to  $f$  and  $g$  by

$$j(\tau) = \frac{(24f)^3}{\Delta}. \tag{A.14}$$

For equation (A.13) to describe 24 branes it should have in general twenty-four distinct solutions. If  $f$  is a polynomial in  $z$  of degree 8 and  $g$  of degree 12, modding out by  $SL(2, \mathbb{C})$  symmetry on the  $z$  plane and the freedom to rescale  $f$  and  $g$ , the number of independent complex parameters is 18. This tells us that there cannot be more than eighteen 7-branes of the same type. This is inevitable since the theory on a compact manifold would have been inconsistent had fluxes not cancelled. It was shown in [7] that type IIB on an orientifold  $T^2/\mathbb{Z}_2$  with four orientifold 7-planes, each coinciding with four D7-branes, is F-theory on K3 at a particular point of the complex moduli space where the modulus  $\tau$  is a constant. The

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<sup>5</sup>There is another complication about the signature of these extra dimensions being (1,1). But as explained in [24] we can treat it as a Euclidean space.



charges of the D7-branes and the O7-planes cancel exactly everywhere and the number of massless gauge neutral fields is exactly 18. They correspond to the modulus  $\tau$ , 16 massless fields that give the D7-brane position and  $4 - 3 = 1$  orientifold positions modulo  $SL(2, \mathbb{C})$  on the  $z$  plane. Analogous considerations for intersecting branes are discussed in the main text.

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